

# STRONGLY ROBUST TORIC IDEALS IN CODIMENSION 2

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**ABSTRACT.** A homogeneous ideal is robust if its universal Gröbner basis is also a minimal generating set. For toric ideals, one has the stronger definition: A toric ideal is strongly robust if its Graver basis equals the set of indispensable binomials. We characterize the codimension 2 strongly robust toric ideals by their Gale diagrams. This give a positive answer to a question of Petrovic, Thoma, and Vladioiu in the case of codimension 2 toric ideals.

## 1. INTRODUCTION

A homogeneous ideal is robust if its universal Gröbner basis is also a minimal generating set. Although one typically expects the universal Gröbner basis to be much larger than a minimal generating set (and hence most ideals are far from robust), there are a surprising number of examples of ideals that are robust. Usually these examples have rich underlying combinatorics. Three well-known example are: the ideals of maximal minors of generic matrices of indeterminates [2, 8], the vanishing ideal of the closure of an affine linear space in  $(\mathbb{P}^1)^n$  [1], and toric ideals of Lawrence type (see [7, Chapter 7]).

Let  $A \in \mathbb{Z}^{d \times n}$  be an integer matrix of rank  $d$ , and  $\mathbb{K}[p] := \mathbb{K}[p_1, \dots, p_n]$  the polynomial ring in  $n$  indeterminates. The toric ideal associated to the matrix  $A$  is the binomial ideal

$$I_A = \langle p^u - p^v : u, v \in \mathbb{N}^n, Au = Av \rangle.$$

Properties of the generating set of  $I_A$  and the geometry of the corresponding variety are determined by combinatorial properties of the matrix  $A$ , and many conditions can be expressed in terms of linear algebra over the integers. Boocher and Robeva [4] initiated a systematic study of robustness of toric ideals and introduced the word “robust”. They showed that a set of quadratic binomials generate a robust ideal if and only if it is the direct sum of ideals of maximal minors of  $2 \times n_i$  generic matrices on disjoint sets of variables. Since these ideals are toric ideals of Lawrence type, one wonders if all robust toric ideals must be of Lawrence type. Petrovic, Thoma, and Valdoiu [6] studied this problem by introducing an oriented matroid concept they call “bouquets”, which we explain below. They also introduced a strengthening of robust for toric ideals, which they called  $\emptyset$ -Lawrence, and we call strongly robust, that involves looking at a superset of the universal Gröbner basis called the Graver basis (explained in Section 2).

Associated to the matrix  $A$  is the Gale transform  $B$  which is a  $n \times n - d$  integer matrix whose columns span  $\ker_{\mathbb{Z}} A$ . When describing the matrix  $A$ , we often think about it as a list of column vectors  $A = \{a_1, a_2, \dots, a_n\}$ . When describing the Gale transform we think about it as a list of row vectors  $B = \{b_1, b_2, \dots, b_n\}$ . A *bouquet* is a maximal subset  $S \subseteq [n]$  such that  $\text{span}(b_s : s \in S)$  is one-dimensional. A bouquet  $S$  is *mixed* if not all

elements  $\{b_s : s \in S\}$  lie in the same orthant. In the language of matroid theory, bouquets correspond to rank one flats of the dual matroid associated to  $A$ .

A key observation of [6] is that the toric ideals of Lawrence type have many mixed bouquets. Recall that if  $A \in \mathbb{Z}^{d \times n}$ , the *Lawrence lifting* of  $A$  is the matrix

$$\Lambda(A) = \begin{pmatrix} A & 0 \\ I & I \end{pmatrix} \in \mathbb{Z}^{(d+n) \times 2n}$$

where  $I$  denotes an  $n \times n$  identity matrix. A toric ideal  $I_C$  is said to be of Lawrence type if it is equal to  $I_{\Lambda(A)}$  for some matrix  $A$ , perhaps after permuting the indeterminates. Note that

$$\ker_{\mathbb{Z}} \Lambda(A) = \{(u, -u) \in \mathbb{Z}^{2n} : u \in \ker_{\mathbb{Z}} A\}.$$

This means that for a toric ideal of Lawrence type, every  $s \in [2n]$  belongs to a mixed bouquet. Petrovic, Thoma, and Valdoiu also show how to use the bouquet structure to produce new examples of strongly robust toric ideals that are not of Lawrence type, and they posed the following question about strongly robust toric ideals.

**Question 1.1.** If  $I_A$  is a strongly robust toric ideal, must  $A$  have a mixed bouquet?

If  $A \in \mathbb{Z}^{d \times n}$  is a toric ideal, with  $d = \text{rank } A$ , then the codimension of  $I_A$  is  $n - d$ . When the codimension of  $I_A$  is one, in which case  $I_A$  is a principal ideal, Question 1.1 is trivial since  $A$  consists of a single bouquet that must be mixed if  $I_A$  is positively graded. We also provide a positive answer to Question 1.1 in the case that  $I_A$  has codimension 2 by giving a complete characterization of the strongly robust codimension 2 toric ideals in terms of the Gale transform, which is described in following sections. One consequence is the following:

**Theorem 1.2.** *Let  $A \in \mathbb{Z}^{(n-2) \times n}$  be a full rank matrix, and  $\tilde{B} = \{b_1, \dots, b_n\} \subseteq \mathbb{Z}^2$  be the reduced Gale transform of  $A$ . If  $I_A$  is a strongly robust toric ideal then  $\text{conv}(\tilde{B})$  is a centrally symmetric polygon.*

In fact, Theorem 1.2 provides a more detailed answer to Question 1.1 in the case of codimension 2 toric ideals.

**Corollary 1.3.** *If a codimension 2 toric ideal  $I_A$  is strongly robust then  $A$  has at least 2 mixed bouquets.*

Both of these results will be a consequence of the general characterization of strongly robust codimension 2 toric ideals that we prove in the next section. The proof uses the Peeva-Sturmfels [5] theory of toric ideals of codimension 2. While the result of Theorem 1.2 does not directly generalize to toric ideals of higher codimension, it does suggest that the property of being strongly robust is connected to the geometry of the Gale transform, which might suggest other approaches to Question 1.1.

## 2. PROOF OF THEOREM 1.2 AND COROLLARY 1.3

To prove Theorem 1.2 we need more details about strongly robust toric ideals, and results about generating sets of codimension 2 toric ideals.

First of all, we need to formally introduce the definition of strongly robust [3, 6]. To explain this we introduce some definitions. Given a vector  $u \in \mathbb{Z}^n$  the support of  $u$ ,  $\text{supp}(u) \subseteq [n]$  is the set of indices  $i$  where  $u_i$  is not zero. Let  $u \in \mathbb{N}^n$ . The fiber of  $u$  is the set  $\mathcal{F}(u) = \{v \in \mathbb{N}^n : Au = Av\}$ . Clearly if  $p^u - p^v \in I_A$  then  $u, v$  belong to the same fiber. A binomial  $p^u - p^v$  is called an *indispensable binomial* if  $\mathcal{F}(u) = \{u, v\}$  and  $\text{supp}(u) \cap \text{supp}(v) = \emptyset$ . The set of all indispensable binomials is denoted  $\mathcal{S}(A)$ . A binomial  $p^u - p^v \in I_A$  is called *primitive* if there is no other binomial  $p^{u'} - p^{v'} \in I_A$  such that  $p^{u'} | p^u$  and  $p^{v'} | p^v$ . The set of all primitive binomials in  $I_A$  is called the *Graver basis* of  $A$ , and denoted  $\mathcal{Gr}(A)$ . The universal Gröbner basis of  $A$  is a subset of the Graver basis, and the set of indispensable binomials are a subset of the universal Gröbner basis. The set of indispensable binomials appear in every binomial minimal generating set of  $I_A$ . This leads to the following definition:

**Definition 2.1.** The toric ideal  $I_A$  is *strongly robust* if  $\mathcal{S}(A) = \mathcal{Gr}(A)$ .

In [6] strongly robust toric ideals are called  $\emptyset$ -Lawrence. Clearly every strongly robust toric ideal is robust. Boocher et al [3] wonder (Question 6.1) if robust implies strongly robust for toric ideals, and prove this is true in some instance associated to graphs.

One useful tool for analyzing the Graver bases of  $I_A$ , is its connection to the Lawrence lifting. Recall that the definition of the Lawrence lifting from the introduction. Its toric ideal  $I_{\Lambda(A)}$  we consider in the ring  $\mathbb{K}[p, q]$ . Binomials in  $I_{\Lambda(A)}$  have the form  $p^u q^v - p^v q^u$  such that  $p^u - p^v \in I_A$ .

**Theorem 2.2.** Let  $A \in \mathbb{Z}^{d \times n}$ . Let  $M = \{p^{u_i} q^{v_i} - p^{v_i} q^{u_i} : i = 1, \dots, m\}$  be a binomial minimal generating set of the toric ideal  $I_{\Lambda(A)}$ . Then  $\{p^{u_i} - p^{v_i} : i = 1, \dots, m\}$  is the Graver basis of  $I_A$ .

A key tool for studying toric ideals in codimension 2 are the reduced Gale diagrams. These were studied by Peeva and Sturmfels [5] to give a complete description of their free resolution. We define them now:

Let  $A \in \mathbb{Z}^{(n-2) \times n}$  be a matrix of rank  $n - 2$  and  $B$  the resulting Gale configuration. Let  $B = \{b_1, \dots, b_n\}$  be the resulting list of row vectors, with  $b_i = (b_{i1}, b_{i2})$ . The *reduced Gale configuration*  $\tilde{B} = \{\tilde{b}_1, \dots, \tilde{b}_n\}$  is obtained by setting

$$\tilde{b}_i = \gcd(b_{i1}, b_{i2})^{-1}(-b_{i2}, b_{i1}).$$

That is,  $\tilde{B}$  is obtained from  $B$  by rotating the vectors by 90 degrees and scaling so that elements in each vector are relatively prime. For the notion of a minimal generating set to be meaningful, we need to assume that the toric ideal  $I_A$  is positively graded. In terms of the reduced Gale configuration, this means that there is no nonzero vector  $w \in \mathbb{Q}^2$  such that  $w^T \tilde{b}_i > 0$  for all  $i$ . With this assumption, the vectors  $\tilde{b}_i$  can be ordered in such a way that each pair  $\tilde{b}_i, \tilde{b}_{i+1}$  span a cone such that no other  $\tilde{b}_j$  lies in the interior of the cone (where  $\tilde{b}_{n+1} = \tilde{b}_1$ ).

For each cone  $\text{cone}(\tilde{b}_i, \tilde{b}_{i+1})$ , let  $H_i$  be its Hilbert basis, which is the minimum generating set of the monoid  $\text{cone}(\tilde{b}_i, \tilde{b}_{i+1}) \cap \mathbb{Z}^2$ . Define the *Hilbert basis* of the reduced Gale configuration to be the set:

$$\mathcal{H}_A = \{u \in \mathbb{Z}^2 : \text{both } u \text{ and } -u \text{ are in } H_1 \cup H_2 \cup \dots \cup H_n\}.$$

**Theorem 2.3.** [5, Theorem 3.7] *Let  $A \in \mathbb{Z}^{(n-2) \times n}$  have rank  $n - 2$ , and  $B$  the Gale configuration. A vector  $u \in \mathbb{Z}^2$  is in  $\mathcal{H}_A$  if and only if  $p^{(Bu)+} - p^{(Bu)-}$  is an indispensable binomial of the toric ideal  $I_A$ . Furthermore, the indispensable binomials are a generating set for  $I_A$ , unless there are no indispensable binomials, in which case  $I_A$  is a complete intersection.*

Hilbert bases are complicated to compute for general cones, but in dimension 2 there is a particularly simple geometric description.

**Proposition 2.4.** *Let  $a, b \in \mathbb{Z}^2$  and let  $P = \text{cone}(a, b)$ . The Hilbert basis of  $P$  consists of all lattice points in the polyhedron  $\text{conv}((P \cap \mathbb{Z}^2) \setminus \{(0, 0)\})$  that are visible from the origin.*

Combining Theorems 2.2 and 2.3, the Graver basis of  $A$  can also be characterized in terms of the reduced Gale configuration.

**Corollary 2.5.** *Let  $A \in \mathbb{Z}^{(n-2) \times n}$  have rank  $n - 2$ , and  $B$  the Gale configuration. Suppose that  $\ker_{\mathbb{Z}} A \cap \mathbb{N}^n = \{0\}$ . A vector  $u \in \mathbb{Z}^2$  has either  $u$  or  $-u \in H_1 \cup \dots \cup H_n$  if and only if  $p^{(Bu)+} - p^{(Bu)-}$  is a primitive binomial of the toric ideal  $I_A$ .*

*Proof.* For the Gale configuration  $B$  of  $A$ , define  $B^{\pm} := B \cup -B$ , which is the Gale configuration of the Lawrence lifting  $\Lambda(A)$ , and let  $\tilde{B}^{\pm}$  be the reduced Gale configuration. As for  $B$ , we assume that the elements of  $\tilde{B}^{\pm}$  are ordered so that each cone  $\text{cone}(\tilde{b}_i, \tilde{b}_{i+1})$  no other  $\tilde{b}_j$  lies in its interior. Let  $H_i^{\pm}$  be the hilbert basis of  $\text{cone}(\tilde{b}_i, \tilde{b}_{i+1})$ . Since  $\tilde{B}^{\pm}$  is centrally symmetric, the hilbert basis of the resulting Lawrence configuration  $\Lambda(A)$  will be the union of all the  $H_i^{\pm}$ . By Theorem 2.3 these vectors determine the minimal generating set of  $I_{\Lambda(A)}$ . By Theorem 2.2 those vectors then determine the Graver basis of  $I_A$ . So to prove the corollary, we need to show that every  $u$  in some  $H_i^{\pm}$ , either  $u$  or  $-u$  appears in  $H_1 \cup \dots \cup H_n$ .

So let  $u \in H_i^{\pm}$ . If  $\tilde{b}_i$  and  $\tilde{b}_{i+1}$  are both in  $B$  or both in  $-B$ , then  $\text{cone}(\tilde{b}_i, \tilde{b}_{i+1})$  or  $\text{cone}(-\tilde{b}_i, -\tilde{b}_{i+1})$  is one of the cones described in the Hilbert basis of  $A$ , so  $u$  or  $-u$  belongs to  $H_1 \cup \dots \cup H_n$ . This leaves the case that  $\tilde{b}_i \in B$  and  $\tilde{b}_{i+1} \in -B$  (the reverse situation follows from a symmetric argument). Looking at the ordering on  $B$ , there will be a unique smallest  $j$  such that  $\tilde{b}_j \in B$  and  $\text{cone}(\tilde{b}_i, \tilde{b}_j)$  forms one of the cones for computing  $H_1 \cup \dots \cup H_n$ . Similarly, there is a unique largest  $k$  such that  $-\tilde{b}_k \in B$  and  $\text{cone}(-\tilde{b}_k, -\tilde{b}_{i+1})$  forms one of the cones for computing  $H_1 \cup \dots \cup H_n$ . These vectors are guaranteed to exist by the positive grading assumption that  $\ker_{\mathbb{Z}} A \cap \mathbb{N}^n = \{0\}$ . Clearly, we have that

$$\text{cone}(\tilde{b}_i, \tilde{b}_{i+1}) = \text{cone}(\tilde{b}_i, \tilde{b}_j) \cap \text{cone}(\tilde{b}_k, \tilde{b}_{i+1}).$$

Furthermore, if we let

$$P_{i,i+1} = \text{conv}(\text{cone}(\tilde{b}_i, \tilde{b}_{i+1}) \cap \mathbb{Z}^2 \setminus \{(0, 0)\})$$

and defined  $P_{i,j}$  and  $P_{k,i+1}$  similarly, then we have that

$$P_{i,i+1} = P_{i,j} \cap P_{k,i+1}.$$

Since each of  $P_{i,i+1}$ ,  $P_{i,j}$ , and  $P_{k,i+1}$  is the convex hull of lattice points, the lattice points visible from the origin in  $P_{i,i+1}$ , will be either a lattice point visible from the origin in  $P_{i,j}$  or in  $P_{k,i+1}$ , or both. In the case at a  $u \in P_{i,i+1}$  is a lattice point visible from the origin with  $u \in P_{i,j}$ , then  $u \in H_1 \cup \dots \cup H_n$ . In the case that  $u \in P_{i,i+1}$  is a lattice point visible from the origin with  $u \in P_{k,i+1}$  then  $-u \in H_1 \cup \dots \cup H_n$ .  $\square$

Corollary 2.5 then reduces the problem of characterizing strongly robust toric ideals in codimension 2 to the following problem.

**Problem 2.6.** For which rank  $n - 2$  matrices  $A \in \mathbb{Z}^{(n-2) \times n}$  is the Hilbert basis  $\mathcal{H}_A$  equal to  $H_1 \cup H_2 \cup \dots \cup H_n$ .

The answer is contained in the following Lemma.

**Lemma 2.7.** Let  $A \in \mathbb{Z}^{(n-2) \times n}$  have rank  $n - 2$  and  $\tilde{B}$  the reduced Gale diagram. Then  $I_A$  is strongly robust if and only if for each  $b_i \in \tilde{B}$ ,  $-b_i \in \mathcal{H}_A$ .

*Proof.* Clearly the condition of the theorem is necessary since  $b_i$  itself always belongs to  $H_1 \cup \dots \cup H_n$ . On the other hand, if  $-b_i \in \mathcal{H}_A$  but  $-b_i \notin B$ , we can add it to  $B$  without changing  $\mathcal{H}_A$ . Indeed, if  $\text{cone}(b_j, b_{j+1})$  contains  $-b_i$  as a visible lattice point of  $P_{j,j+1}$ , then the visible lattice points arising in the cones  $\text{cone}(b_j, -b_i)$  and  $\text{cone}(-b_i, b_{j+1})$  are precisely the visible lattice points in  $P_{j,j+1}$ . By repeating this procedure, we end up with a Gale diagram that contains only pairs  $b_i, -b_i$ , which is the Gale diagram of a Lawrence matrix. Hence,  $I_A$  is strongly robust.  $\square$

Now we are in a position to Prove Theorem 1.2.

*Proof of Theorem 1.2.* Suppose that  $I_A$  is a strongly robust toric ideal in codimension 2, let  $\tilde{B}$  be the reduced Gale configuration, and  $P = \text{conv}(\tilde{B})$  the convex hull of the elements in  $\tilde{B}$ . Let  $\tilde{b}$  be a vertex of  $P$ . We must show that  $-\tilde{b}$  is also a vertex of  $P$  to see that  $P$  is centrally symmetric.

Since  $\tilde{b} \in B$ , and  $I_A$  is strongly robust,  $-\tilde{b}$  belongs to  $\mathcal{H}_A$ , by Lemma 2.7. If  $-\tilde{b}$  is not a vertex of  $P$ , then there are two vectors  $b_1, b_2 \in \tilde{B}$  such that  $-\tilde{b}$  is in  $\text{conv}(b_1, b_2, (0, 0))$ . Applying Lemma 2.7 again, we have the  $-b_1$  and  $-b_2$  are in  $\mathcal{H}_A$ . In particular, these two vectors are in  $P$ , by Proposition 2.4. However, this forces that  $\tilde{b} \in \text{conv}(-b_1, -b_2, (0, 0))$ , so  $\tilde{b}$  could not be a vertex of  $P$ .  $\square$

*Proof of Corollary 1.3.* Since the polytope  $\text{conv}(\tilde{B})$  must be two dimensional and is centrally symmetric, it must have at least two pairs of opposite vertices  $\tilde{b}_1, -\tilde{b}_1$  and  $\tilde{b}_2, -\tilde{b}_2$ . These two pairs of opposite vertices yield two mixed bouquets of the matrix  $A$ .  $\square$

**Example 2.8.** Let  $A$  be the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ -2 & 0 & 0 & 0 & -4 & 5 \end{pmatrix}$$

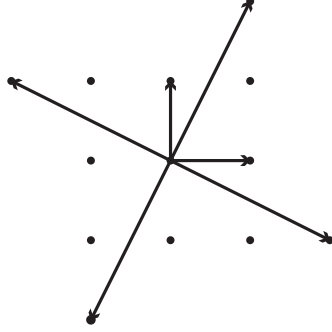


FIGURE 1. The reduced Gale transform of the matrix in Example 2.8. The dots represent the points in the set  $\mathcal{H}_A$ .

which has the Gale transform  $B$  and reduced Gale transform  $\tilde{B}$  respectively:

$$B = \begin{pmatrix} 1 & 2 \\ -2 & 1 \\ -1 & -2 \\ 0 & -1 \\ 2 & -1 \\ 2 & 0 \end{pmatrix} \quad \tilde{B} = \begin{pmatrix} -2 & 1 \\ -1 & -2 \\ 2 & -1 \\ 1 & 0 \\ 1 & 2 \\ 0 & 1 \end{pmatrix}$$

As can be seen from the reduced Gale transform, illustrated in Figure 1, the matrix  $A$  satisfies the condition of Lemma 2.7, and so the toric ideal is strongly robust. The minimal generating set, which equals the Graver basis, consists of the following 6 binomials that are in bijection with pairs of opposite lattice points in  $\mathcal{H}_A$ .

$$I_A = \langle b^5 - de^5f^4, ae^2f^2 - b^2c, ab^3 - cde^3f^2, a^2b - c^2de, a^3ef^2 - bc^3d, a^5f^2 - c^5d^2 \rangle$$

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